

A classical appraisal of quantum definitions of non-Markovian dynamics

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Abstract. We consider the issue of non-Markovianity of a quantum dynamics starting from a comparison with the classical definition of Markovian process. We point to the fact that two sufficient but not necessary signatures of non-Markovianity of a classical process find their natural quantum counterpart in recently introduced measures of quantum non-Markovianity. This behavior is analyzed in detail for quantum dynamics which can be built taking as input a class of classical processes.

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1. Introduction

In the field of open quantum systems the interaction with an external environment introduces a stochasticity element in the dynamics, which has typically been described as a quantum process in analogy with classical processes [1]. Quantum Markovian processes have been understood as described by a master equation in Lindblad form [2, 3], which grants complete positivity. Furthermore these dynamics can be understood as an average over trajectories corresponding to suitable measurements continuous in time on the side of the environment, thus providing a direct link with the description of quantum measurement [4].

However the very notion of non-Markovianity for a quantum dynamics has long remained vague, being usually associated with the occurrence of revivals or non exponential relaxation, apart from mathematical work in which a proper definition of a quantum process has been addressed [5, 6]. Most recently, different approaches have been devised in order to assess and quantify non-Markovianity in the dynamics of an open quantum system, looking at states rather than at multitime correlation functions. One approach, based on the idea that memory can be seen as a backflow of information from the environment to the system, studies the distinguishability among states in the course of the dynamics, identifying non-Markovianity with revivals in the distinguishability. The amount of non-Markovianity then depends on frequency and relevance of these revivals [7]. Note that this approach is actually valid in order to

estimate non-Markovianity of the dynamics in any statistical theory, simply considering the mathematical representation of the set of states and a distance on it, which should be a contraction under the action of positive probability preserving maps. Another approach relies on the divisibility over arbitrary intermediate time intervals of the evolution map into well defined maps preserving positivity and probability. It estimates non-Markovianity quantifying the failure of this property [8].

In recent work it has been pointed out that these two approaches can be connected to certain signatures of non-Markovianity to be detected at the level of the one-point probability in a stochastic process [9], and a few examples of this behavior have been considered building on a class of stochastic processes known as semi-Markov processes [10, 11]. In this contribution we will consider the class of maps which can be built within this framework by considering quantum dynamics determined by a generic unital stochastic map and an arbitrary waiting time distribution. We will exploit these examples to study in detail the different performance of the two measures and the variety of possible behavior which can be obtained. This class of dynamics will also allow us to point to the proper relationship between divisibility of the time evolution in terms of completely positive maps and behavior of the coefficients in the time-convolutionless form of the equations of motion.

The paper is organized as follows. In Sect. 2 we introduce the definition of classical Markov process, together with previous proposals to extend this definition to the quantum realm. We further consider two sufficient signatures of non-Markovianity to be read from the one-point probability density. In Sect. 3 it is shown how the extension of these signatures to the quantum case recovers two recent proposals for the definition of a non-Markovian dynamics, which are studied and compared for a general class of examples. Conclusions and final remarks are presented in Sect. 4.

2. Markovian and non-Markovian processes

The notion of stochastic process is of the greatest importance in the description of phenomena in which randomness appears. Due to interaction with the environment open quantum systems do provide a natural setting in which a stochastic description appears. This in turns leads to the difficult question about how to properly characterize a quantum stochastic process, addressed in different ways by physicists [12, 1] and mathematicians [5, 6]. Within this framework a further important question is how to define Markovianity of a process and how to associate a memory to the various phenomena.

In the classical case one has a precise definition of Markovian process, relying on the knowledge of all finite dimensional distributions of the process, which is best formulated in terms of conditional probabilities [13]

$$p_{1|n}(x_n, t_n | x_{n-1}, t_{n-1}; \dots; x_0, t_0) = p_{1|1}(x_n, t_n | x_{n-1}, t_{n-1}) \quad (1)$$

with $t_n \geq t_{n-1} \geq \dots \geq t_1 \geq t_0$. It tells that the probability for the random variable to assume the value x_n at time t_n only depends on the last assumed value, and not

on previous ones, thus properly formalizing the notion of lack of memory. The set of conditions Eq. (1) lead in particular to the Chapman-Kolmogorov equation obeyed by the two point conditional probability, also called propagator

$$p_{1|1}(x, t|y, s) = \sum_z p_{1|1}(x, t|z, \tau) p_{1|1}(z, \tau|y, s), \quad (2)$$

valid for $t \geq \tau \geq s$. A solution of this equation fully characterizes a Markovian process in that all finite dimensional distributions can be obtained given the initial probability distribution. Since the propagator is on its turn determined from the dynamics of all mean values, this means that the whole process is known from the dynamics of mean values, a result known as regression theorem [14]. All multitime correlation functions of the process are thus fixed from the propagator and therefore from the mean values. Indeed Lindblad introduced a definition of quantum Markovian process by relying on the validity of the quantum regression theorem [5]. Consider a system S interacting with an environment E according to a unitary evolution $U(t)$ and multitime correlation functions of the form

$$\langle \mathcal{M}_n(t_n) \dots \mathcal{M}_1(t_1) \rangle = \text{Tr}_S \text{Tr}_E \{ \mathcal{M}_n U(t_n - t_{n-1}) \mathcal{M}_{n-1} \dots \mathcal{M}_1 U(t_1) \rho_S \otimes \rho_E \},$$

where \mathcal{M}_i denotes a quantum operation corresponding to measurement of a certain system observable, e.g. $\mathcal{M}(\varrho) = \sum_k P_k \varrho P_k$ for the von Neumann measurement of the self-adjoint operator $A = \sum_k a_k P_k$. If the quantum regression formula applies such multitime correlation functions can be expressed as

$$\langle \mathcal{M}_n(t_n) \dots \mathcal{M}_1(t_1) \rangle = \text{Tr}_S \{ \mathcal{M}_n \Phi(t_n - t_{n-1}) \mathcal{M}_{n-1} \dots \mathcal{M}_1 \Phi(t_1) \rho_S \},$$

with $\Phi(t)$ completely positive maps acting on the system only. In the weak and singular coupling limit these maps can be shown to satisfying a semigroup composition law [15], so that they can be expressed with a generator in Lindblad form. As a result in particular mean values and correlations do obey the same dynamical equations. This result, though by no means obvious for the behavior of the reduced dynamics of an open system, is a basic working tool in open quantum system theory [1], and the extension of the validity of the regression formula to the non-Markovian setting is an issue of great relevance. The problem has been recently addressed for a special class of non-Markovian evolutions known as generalized Lindblad structure [16, 17, 18], showing however that regression formula are generally not valid [19].

2.1. Witnesses and quantifiers of non-Markovianity

The validity of the Chapman-Kolmogorov equation for a Markovian process entails two simple consequences on the behavior of the propagator, which are at the heart of two recently introduced quantum measures of non-Markovianity [7, 8]. Let us consider for the sake of simplicity a random variable taking values on a finite set, so that the propagator from time s to time t can be written as a stochastic matrix $\Lambda(t, s)$ and the state of the system at a given time t can be expressed as a probability vector $\mathbf{p}(t)$. As

distance among probability vectors we consider the Kolmogorov distance [20]

$$D_K(\mathbf{p}^1(t), \mathbf{p}^2(t)) = \frac{1}{2} \sum_i |p_i^1(t) - p_i^2(t)|.$$

Upon validity of the Chapman-Kolmogorov equation one has $\mathbf{p}(t) = \Lambda(t, s)\mathbf{p}(s)$ for any $0 \leq s \leq t$, and exploiting the definition of stochastic matrix this entails that the Kolmogorov distance among two probability distributions following a Markovian dynamics do decrease monotonously in time

$$D_K(\mathbf{p}^1(t), \mathbf{p}^2(t)) \leq D_K(\mathbf{p}^1(s), \mathbf{p}^2(s)) \quad \forall t \geq s.$$

This relation naturally provides a necessary condition for the Markovianity of a stochastic process, stating that in a Markovian process the one-time probability densities or distribution functions evolving from two distinct initial situations do get less and less distinguishable. A non monotonic behavior in time of the Kolmogorov distance among two states thus provides a witness of the non-Markovianity of the process. Of course this condition now appears as a sufficient condition only, or if one prefers a different definition of Markovianity.

Let us stress that the Kolmogorov distance actually corresponds to the L_1 distance among probability vectors, thus being a natural distance in any statistical theory. Indeed a statistical theory relies on the existence of two spaces, one dual to the other, in which states and observables do live. The specific choice of spaces and their commutativity or non commutativity do fix the statistical structure of the theory [21, 22, 23, 24]. In the case of classical mechanics one considers the L_1 space of probability distributions on phase space, while the observables are given by the L_∞ space of bounded functions. Considering quantum mechanics the dual Banach spaces are given by the space of trace-class operators with the trace norm topology, in which states are described by statistical operators, and the space of bounded operators with the uniform norm, in which to consider the observables of the theory. The Kolmogorov distance thus naturally becomes the trace distance among statistical operators.

The validity of the Chapman-Kolmogorov equation for a Markov process entails another important consequence at the level of the one-point probability density, namely rewriting Eq. (2) in terms of stochastic matrices for a finite dimensional process one has $\Lambda(t, s) = \Lambda(t, \tau)\Lambda(\tau, s)$ for any $s \leq \tau \leq t$. Taking s as an initial time set equal to zero one has the relation

$$\Lambda(t, 0) = \Lambda(t, \tau)\Lambda(\tau, 0) \tag{3}$$

where the crucial fact is that each Λ is a well-defined stochastic matrix sending any probability vector to a probability vector. For a Markov process these matrices are fixed by the transition probabilities. Eq. (3) then describes a divisibility property which provides a witness of Markovianity. However the violation of this property only provides a necessary but not sufficient condition in order to ascertain Markovianity. Indeed also for non-Markovian processes one might find a collection of stochastic matrices obeying Eq. (3), which nevertheless do not coincide with the transition probabilities of the

process [25]. Otherwise stated these collection of stochastic matrices can be taken as the transition probabilities of a Markov process which at the level of the one-point probabilities cannot be distinguished from the original one.

These signatures of non-Markovianity for a classical process correspond to criteria used to define a non-Markovian dynamics in the quantum case in [7] and [8], by considering distinguishability as quantified by the trace distance, and divisibility as characterized by a composition in terms of completely positive maps respectively. In order to further assign a weight to the deviation from the Markovian behavior, relying on these signatures two measures of non-Markovianity have been introduced, which essentially assign a weight to the time intervals in which either trace distance grows or divisibility, at the quantum level to be understood as divisibility in terms of completely positive maps, fails. These weights can actually be assigned in different ways. For example one can consider quantifiers of distinguishability on the state space other than trace distance, provided they are contractions with respect to the action of positive maps [26, 27], e.g. relative entropy which appears in a natural way in certain dissipative systems [28, 29].

2.2. Semi-Markov processes

Let us now consider a class of stochastic processes for which a simple characterization is available, and which include both Markovian and non-Markovian processes, namely semi-Markov processes [30]. Such processes arise considering transitions among different sites determined from certain jump probabilities, fixed by a stochastic matrix as in a Markovian chain, and random waiting times between jumps determined by site dependent waiting time distributions. They are characterized via a semi-Markov matrix

$$(Q)_{mn}(\tau) = (\Pi)_{mn}f_n(\tau), \quad (4)$$

whose entries give the probability density for a jump from site n to site m in a given time τ . The transition probabilities π_{mn} build up a stochastic matrix, while the $f_n(\tau)$ are the waiting time distributions which actually characterize whether the process is Markovian or not according to the classical definition. Indeed such a process is Markovian only if all waiting time distributions are memoryless, that is exponentially distributed

$$f_n(\tau) = \lambda_n e^{-\lambda_n \tau}. \quad (5)$$

In all other cases such processes are non-Markovian. This simple characterization of semi-Markov processes, together with the fact that their transition probabilities can be obtained as solution of a generalized master equation with a memory kernel, allows to exemplify the meaning of the signatures of non-Markovianity introduced above. In the case in which all waiting time distributions are equal, though otherwise arbitrary, the transition probability $T(t, 0)$ of such a process obeys a closed integrodifferential equation given by

$$\frac{d}{dt}T(t, 0) = \int_0^t d\tau (\Pi - \mathbb{1})k(\tau)T(t - \tau, 0), \quad (6)$$

where $k(\tau)$ is a memory kernel fixed by $f(\tau)$ according to the relationship

$$f(\tau) = \int_0^\tau dt k(\tau - t)g(t), \quad g(t) = 1 - \int_0^t d\tau f(\tau), \quad (7)$$

where $g(t)$ is the survival probability associated to $f(\tau)$.

We now consider a two-dimensional system and take two equal waiting time distributions, so that the semi-Markov matrix of Eq. (4) can be written

$$Q(\tau) = \begin{pmatrix} \pi & \sigma \\ 1 - \pi & 1 - \sigma \end{pmatrix} f(\tau), \quad (8)$$

with π, σ the jump probabilities from one site to the other. Starting from Eq. (6) for the case $\pi = \sigma = 1/2$, one can express the transition probability as a function of the survival probability only, and using $T(t, s) = T(t, 0)T^{-1}(s, 0)$ one obtains the matrices

$$T(t, s) = \frac{1}{2} \begin{pmatrix} 1 + g(t)/g(s) & 1 - g(t)/g(s) \\ 1 - g(t)/g(s) & 1 + g(t)/g(s) \end{pmatrix}, \quad (9)$$

which connect probability vectors at a time s with probability vectors at a later time t . Since g is a survival probability, these matrices are stochastic matrices for any couple $t \geq s$, independently on the fact that the associated semi-Markov process is Markovian only if f is exponentially distributed. Thus for any choice of $f(\tau)$ different from the exponential one has an example of process which is non-Markovian in the classical sense, but still whose one-point probabilities do contract in a monotonous way with respect to the Kolmogorov distance and are connected by stochastic matrices. This indeed shows that contractivity under the Kolmogorov distance and divisibility are sufficient but not necessary criteria in order to detect non-Markovianity in the classical sense.

As a complementary situation, let us consider the case $\pi = 0, \sigma = 1$, so that once in a state the system jumps with certainty to the other, thus obtaining

$$T(t, s) = \frac{1}{2} \begin{pmatrix} 1 + q(t)/q(s) & 1 - q(t)/q(s) \\ 1 - q(t)/q(s) & 1 + q(t)/q(s) \end{pmatrix}. \quad (10)$$

The role of the survival probability $g(t)$ is here replaced by the quantity $q(t)$ whose Laplace transform reads

$$\hat{q}(u) = \frac{1}{u} \frac{1 - \hat{f}(u)}{1 + \hat{f}(u)}.$$

Recalling that the probability for n jumps in a time t for a waiting time distribution $f(t)$ is given by

$$p_n(t) = \int_0^t d\tau f(t - \tau) p_{n-1}(\tau), \quad (11)$$

so that $\hat{p}_n(u) = \hat{p}_0(u) \hat{f}^n(u)$, one finally has

$$q(t) = \sum_{n=0}^{\infty} p_{2n}(t) - \sum_{n=0}^{\infty} p_{2n+1}(t). \quad (12)$$

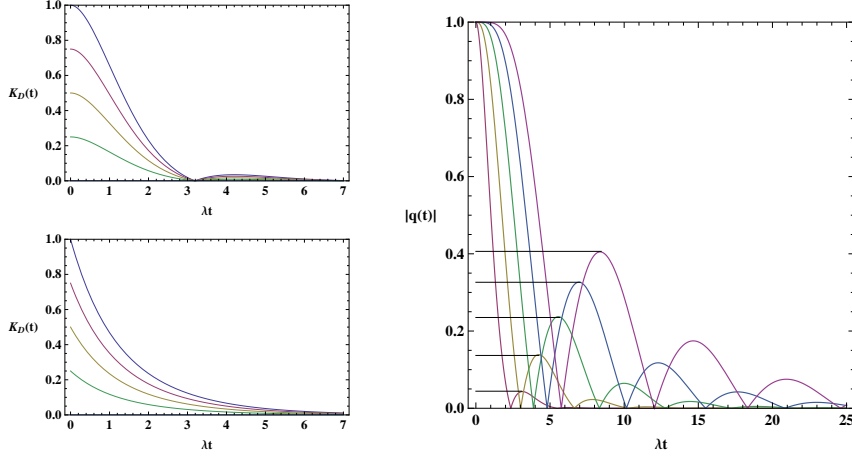


Figure 1. (Color online) (top left) Kolmogorov distance for a classical semi-Markov process characterized by $\pi = 0$ and $\sigma = 1$, and a waiting time distribution given by the convolution of two exponential ones with $\lambda_2 = 0.5\lambda_1$. The different trajectories correspond to initial probability vectors such that $|p_1^1(0) - p_1^2(0)|$ are equally distributed between 0 and 1. Revivals clearly appear. (bottom left) Kolmogorov distance for a semi-Markov process with the same waiting time distribution but $\pi = \sigma = 1/2$. The distance decreases monotonically. In both cases the process is non Markovian according to the classical definition. (right) The modulus of the functions q_m , corresponding to the convolution of m equal exponential distributions, for m up to 6 and in the case of a semi-Markov process with $\pi = 0$ and $\sigma = 1$. The horizontal lines denote the height of the first non trivial maximum of $|q_m(t)|$, quantifying the non-Markovianity of the first interval in which trace distance grows.

The quantity $q(t)$ therefore expresses the difference between the probability to have an even or an odd number of jumps. It immediately appears that this quantity is not necessarily positive, so that the matrices defined by Eq. (10) are not necessarily stochastic matrices, and apart from the case of an exponential waiting time distribution, corresponding to a truly Markovian process, different expression of $f(\tau)$ may or may not lead to contractivity under the Kolmogorov distance and divisibility. These possible behaviors are considered in Fig. 1, where the Kolmogorov distance for two non-Markovian classical process is considered.

3. Quantum non-Markovian dynamics

The characterization of well-defined classes of quantum time evolutions which have the property of being completely positive, though not in Lindblad form, is a highly non trivial task. A whole class of completely positive dynamics can be obtained considering a quantum generalization of the classical semi-Markov processes introduced in Sect. 2.2, also allowing for the connection and comparison with a classical process. Such dynamics are given by the solution of integrodifferential equations with a memory kernel which is formally of Lindblad type [10, 11]. For the case in which the dynamics allows for a

clearcut probabilistic reading such equations can be written in the form [16]

$$\frac{d}{dt}\rho(t) = \int_0^t d\tau k(t-\tau)[\mathcal{E} - \mathbb{1}]\rho(\tau). \quad (13)$$

Indeed if the function $k(t)$ can be associated to a well defined waiting time distribution $f(t)$ according to the relation Eq. (7), then the time evolution of the solution $\rho(t)$ of Eq. (13) can be expressed as repeated actions of the completely positive and trace preserving map \mathcal{E} , also called stochastic map, distributed in time according to the renewal process fixed by $f(t)$

$$\Lambda(t, 0)\rho(0) = \sum_{n=0}^{\infty} p_n(t)\mathcal{E}^n\rho(0), \quad (14)$$

where $p_n(t)$ is defined in Eq. (11). Note that Markovianity or non-Markovianity of this quantum time evolution, according to either criterion of Sect. 2.1, at variance with the classical definition depend on both elements of the couple $\{\mathcal{E}, f(t)\}$. More precisely, while the only truly non-Markovian waiting time distribution corresponding to the exponential leads to a delta correlated kernel in Eq. (13), and therefore to a Lindblad equation, a generic waiting time distribution might still lead to a Markovian dynamics depending on the completely positive trace preserving map \mathcal{E} as we shall see below.

3.1. Unital stochastic maps

Let us first leave $f(t)$ unspecified and consider as \mathcal{E} the so-called Pauli channel

$$\mathcal{E}[\rho] = \lambda_0\rho + \lambda_x\sigma_x\rho\sigma_x + \lambda_y\sigma_y\rho\sigma_y + \lambda_z\sigma_z\rho\sigma_z,$$

with $\boldsymbol{\lambda} \equiv (\lambda_0, \lambda_x, \lambda_y, \lambda_z)$ a probability distribution, $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$. As discussed in [31] up to unitary transformations this expression provides the most general unital stochastic map, which preserves both trace and identity for $\mathcal{H} = \mathbb{C}^2$. The map $\Lambda(t, 0)$ associated to \mathcal{E} through Eq. (14) can be expressed in terms of a basis of superoperators acting on operators in \mathbb{C}^2 . The standard representation is

$$\Lambda[\rho] = \sum_{k,l=0}^3 \Lambda_{kl}\tau_k Tr_S[\tau_l^\dagger \rho], \quad (15)$$

with $\{\tau_k\}_{k=0,1,2,3}$ a basis of operators on \mathbb{C}^2 and $\Lambda_{kl} = Tr_S\{\tau_k^\dagger \Lambda[\tau_l]\}$. The most convenient choice of orthonormal basis corresponds to $\tau_i = \frac{1}{\sqrt{2}}\sigma_i$, with $\sigma_0 = \mathbb{1}$ and σ_i the standard Pauli matrices. These basis elements are eigenoperator of the Pauli channel. The eigenvalues are given by $\boldsymbol{\mu} = A\boldsymbol{\lambda}$, with $A_{0i} = A_{i0} = 1$ and $A_{jk} = 2\delta_{jk} - 1$ for $j, k = 1, 2, 3$, so that $\mu_0 = 1$ and $-1 \leq \mu_i \leq 1$. In order to determine the map $\Lambda(t, 0)$ we consider its action on the basis elements, given by

$$\Lambda(t, 0)[\sigma_i] = \lambda_i(t)\sigma_i, \quad (16)$$

where the quantities $\lambda_i(t)$ are defined as

$$\lambda_i(t) = \sum_{n=0}^{\infty} p_n(t)\mu_i^n, \quad (17)$$

and thus correspond to the generating function of the discrete probability distribution $p_n(t)$ evaluated at μ_i , in particular $\lambda_0(t) = 1$, while $-1 \leq \lambda_i(t) \leq 1$. Note that according to Eq. (16) the standard representation of statistical operators on the Bloch sphere the action of the map $\Lambda(t, 0)$ transforms the surface of the sphere into ellipsoids whose axes have a time dependent length $|\lambda_i(t)|$. In the classical setting Markovianity of the semi-Markov process is only obtained for an exponential waiting time distribution, so that the $p_n(t)$ are a Poisson distribution. In this case the generating function reads

$$\lambda_i(t) = e^{-(1-\mu_i)\lambda t}, \quad (18)$$

where λ is the parameter of the exponential distribution, thus corresponding to ellipsoids whose axes shrink monotonously according to an exponential law. Exploiting Eq. (11) the Laplace transform of the $\lambda_i(t)$ can be directly expressed in terms of the Laplace transform of the waiting time distribution according to

$$\hat{\lambda}_{\mu_i}(u) = \frac{1}{u} \frac{1 - \hat{f}(u)}{1 - \mu_i \hat{f}(u)}. \quad (19)$$

By linearity Eq. (16) determines the map, which is fixed by the matrix elements

$$\begin{aligned} \rho_{11}(t) &= \frac{1}{2} [1 + \lambda_z(t) (\rho_{11}(0) - \rho_{00}(0))] \\ \rho_{10}(t) &= \frac{1}{2} [\rho_{10}(0) (\lambda_x(t) - \lambda_y(t)) + \rho_{01}(0) (\lambda_x(t) + \lambda_y(t))]. \end{aligned}$$

This provides the general solution of Eq. (13) for arbitrary $f(t)$ and \mathcal{E} a generic unital stochastic map. For later use it is also convenient to express the map in a different superoperator basis

$$\Lambda[\rho] = \sum_{k,l=0}^3 \Lambda'_{kl} \tau_k w \tau_l^\dagger, \quad (20)$$

with $\Lambda'_{kl} = \sum_i \text{Tr}_S \{ \tau_l \tau_i^\dagger \tau_k \Lambda[\tau_i] \}$. This representation for superoperators was introduced in [32], and apart from a multiplicative factor corresponding to the space dimension it associates to the map its Choi matrix. For the previous choice of basis set the associated matrix is still diagonal, so that

$$\Lambda(t, 0)[\rho] = \mu_0(t) \rho + \mu_x(t) \sigma_x \rho \sigma_x + \mu_y(t) \sigma_y \rho \sigma_y + \mu_z(t) \sigma_z \rho \sigma_z, \quad (21)$$

with $\boldsymbol{\mu}(t) = A^{-1} \boldsymbol{\lambda}(t) = \frac{1}{4} A \boldsymbol{\lambda}(t)$ and the $\{\mu_i(t)\}$ at any time a probability distribution.

3.2. Analysis of non-Markovianity

We now study Markovianity of these quantum maps according to the criteria introduced in [7, 8]. The first criterion is based on the behavior of the trace distance among distinct initial states, which quantifies how their distinguishability varies in time. For the considered map the trace distance reads

$$\begin{aligned} D(\rho^1(t), \rho^2(t)) &= \frac{1}{2} \|\rho^1(t) - \rho^2(t)\|_1 \\ &= \sqrt{\lambda_z^2(t) \Delta_p(0)^2 + \lambda_x^2(t) \Re \Delta_c(0)^2 + \lambda_y^2(t) \Im \Delta_c(0)^2}, \end{aligned}$$

where $\Delta_p(0)$ and $\Delta_c(0)$ denote population and coherence differences at the initial time. The time derivative of this quantity, which detects non-Markovianity identified with growth of trace distance in time for at least a couple of possible initial states, is

$$\sigma(t, \rho^{1,2}(0)) = \frac{\lambda_z(t) \dot{\lambda}_z(t) \Delta_p(0)^2 + \lambda_x(t) \dot{\lambda}_x(t) \Re \Delta_c(0)^2 + \lambda_y(t) \dot{\lambda}_y(t) \Im \Delta_c(0)^2}{\sqrt{\lambda_z^2(t) \Delta_p(0)^2 + \lambda_x^2(t) \Re \Delta_c(0)^2 + \lambda_y^2(t) \Im \Delta_c(0)^2}},$$

so that a necessary and sufficient condition for non-Markovianity is that at least one of the functions $|\lambda_i(t)|$ grows in a certain time interval. This condition coincides with the requirement of divisibility of the quantum map in terms of positive (but not necessarily completely positive) maps according to Eq. (3). For this time evolution the map $\Lambda(t, s)$ is represented according to Eq. (15) by the matrix

$$\Lambda_{kl}(t, s) = \text{diag} \left(\mathbb{1}, \frac{\lambda_x(t)}{\lambda_x(s)}, \frac{\lambda_y(t)}{\lambda_y(s)}, \frac{\lambda_z(t)}{\lambda_z(s)} \right),$$

which corresponds to a positive map if and only if all eigenvalues lie between zero and one. Let us now spell out these general relations for specific choices of $\{\mathcal{E}, f(t)\}$.

For $\boldsymbol{\lambda} = (0, 0, 0, 1)$ we recover the phase-flip channel

$$\mathcal{E}_z = \sigma_z \rho \sigma_z,$$

which does not have a classical counterpart. For this case one has $\boldsymbol{\mu} = (1, -1, -1, 1)$, leading to $\boldsymbol{\lambda}(t) = (1, q(t), q(t), 1)$, with $q(t)$ as in Eq. (12). The map $\Lambda(t, 0)$ describes pure dephasing, with coherences multiplied by a factor $q(t)$. As a result the time derivative of the quantifier of the distinguishability among the two evolved states grows whenever $|q(t)|$ grows in time. In such time intervals, which we denote collectively by $\Omega_+ = \bigcup_i (a_i, b_i)$, we have

$$\sigma(t, \rho^{1,2}(0)) \leq |\Delta_c(0)| \frac{d}{dt} |q(t)|$$

so that the maximal growth is obtained for opposite equatorial states on the Bloch sphere. The measure of non-Markovianity based on trace distance is then given by

$$\mathcal{N}(\Lambda) = \int_{\Omega_+} dt \frac{d}{dt} |q(t)| = \sum_i (|q(b_i)| - |q(a_i)|).$$

To consider an interesting class of situations we now specify also the waiting time distribution, considering the convolution of m exponential waiting time distributions leading to so called Erlang distributions of the form $f_m(t) = \lambda \frac{(\lambda t)^{m-1}}{(m-1)!} e^{-\lambda t}$, for which the difference of the probabilities to have an even and an odd number of jumps becomes

$$q_m(t) = e^{-\lambda t} \sum_{n=0}^{\infty} (-1)^n \sum_{k=0}^{m-1} \frac{(\lambda t)^{mn+k}}{(mn+k)!}.$$

For the memoryless case $m = 1$ one has the strictly monotone decreasing function $q_1(t) = \exp(-2\lambda t)$ according to Eq. (18), while for $m = 2$ one obtains

$$q_2(t) = e^{-\lambda t} [\cos(\lambda t) + \sin(\lambda t)],$$

which oscillates and crosses zero at isolated points. For $m \geq 2$ these functions exhibit an oscillating behavior, so that the minima of $|q_m(t)|$ lie on the real axis. The modulus

of these functions for values of m up to six is shown in Fig. 1. For each choice of waiting time distribution $f_m(t)$ the measure of non-Markovianity is given by the series

$$\mathcal{N}(\Lambda_m) = \sum_{t_i \in M} |q_m(t_i)|, \quad (22)$$

where M is the denumerable set of times corresponding to the maxima of the function $|q_m|$. An exact analytic evaluation of the measure is feasible for $m = 2$, since the maxima correspond to $\lambda t = n\pi$, with n a positive integer, so that

$$\mathcal{N}(\Lambda_2) = \sum_{n=1}^{\infty} e^{-n\pi} = \frac{1}{e^{\pi} - 1}.$$

More generally as shown in Fig. 1 the first maximum of $|q_m|$ is above the first maximum of $|q_n|$ for $m > n$, and the same occurs for the other maxima, whose values decrease exponentially. This substantiates the statement that waiting time distributions corresponding to the convolution of a higher number of exponential terms have stronger memory [33]. Indeed in such a case the overall waiting time is the sum of a number m of waiting times, which can be thought in series. While each of them is described by a memoryless probability distribution, the overall distribution deviates from the memoryless case, the more so the higher the number of terms. Another situation in which Markovianity or non-Markovianity can be observed is given considering the convolution of different exponential waiting time distributions. In this case we have $f = f_1 * f_2$, where each f_i is of the form Eq. (5) with parameter λ_i , which leads to

$$f(t) = 2 \frac{p}{s} e^{-\frac{1}{2}st} \frac{1}{\sqrt{1 - 4\frac{p}{s^2}}} \sinh\left(\frac{st}{2} \sqrt{1 - 4\frac{p}{s^2}}\right) \quad (23)$$

where s, p denote sum and product of the parameters λ_i . Setting $\lambda_1 = \lambda$ and $\lambda_2 = r\lambda$, so that the parameter $r = \lambda_2/\lambda_1$ gives the relative scale among the rates of the two waiting time distributions the function $q(t)$ takes the

$$q(t) = e^{-\frac{1+r}{2}\lambda t} \left[\cosh\left(\sqrt{r^2 - 6r + 1} \frac{\lambda t}{2}\right) + \frac{1+r}{\sqrt{r^2 - 6r + 1}} \sinh\left(\sqrt{r^2 - 6r + 1} \frac{\lambda t}{2}\right) \right], \quad (24)$$

which can oscillate and take on negative values, thus leading to a non zero measure of non-Markovianity, for $3 - 2\sqrt{2} \leq r \leq 1/(3 - 2\sqrt{2})$. Also in this case non-Markovianity does not appear when one of the rates is much stronger than the other, so that the overall distribution is dominated by only one of the two components, which is memoryless distributed. The sign of $q(t)$ as a function of the relative strength r and the rescaled time λt is plotted in Fig. 2. Note that also in this case the measure of non-Markovianity is given by the sum of the maxima of $|q(t)|$.

The value $\lambda = (0, \frac{1}{2}, \frac{1}{2}, 0)$ leads to the map

$$\mathcal{E}_p \rho = \frac{1}{2} \sigma_x \rho \sigma_x + \frac{1}{2} \sigma_y \rho \sigma_y = \sigma_+ \rho \sigma_- + \sigma_- \rho \sigma_+,$$

for which also populations are affected, since $\mu = (1, 0, 0, -1)$ and $\lambda = (1, g(t), g(t), q(t))$, with $g(t)$ survival probability as in Eq. (7). As discussed in [9] the measure of non-Markovianity of this map according to the trace distance criterion

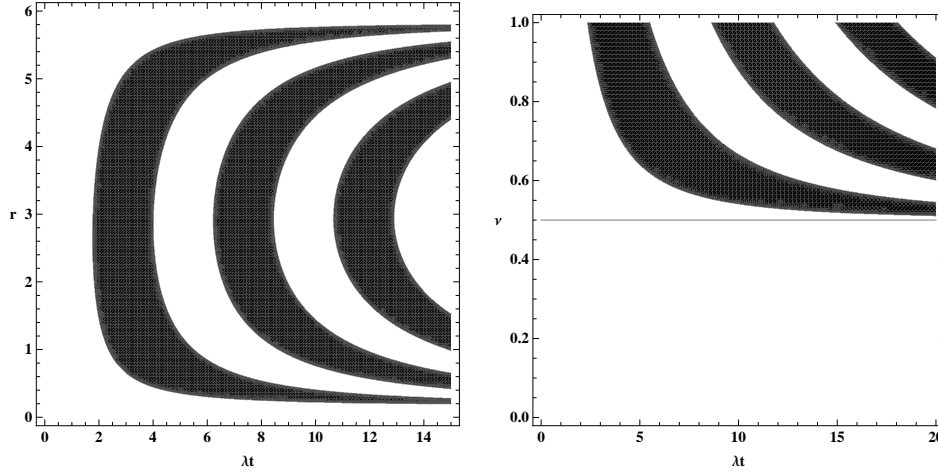


Figure 2. (Color online) (left) Plot of the sign of $q(t)$ given by Eq. (24) for the convolution of two exponential distributions with $\lambda_1 = \lambda$ and $\lambda_2 = r\lambda$ as a function of r and λt . The black regions correspond to a negative sign. (right) Plot of the sign of $\lambda_{1-2\nu}(t)$ for the same waiting time distribution as a function of ν and λt . One clearly sees the threshold for the inset of non-Markovianity at $\nu = 1/2$.

is the same as for the dephasing map characterized by $\boldsymbol{\lambda} = (0, 0, 0, 1)$, so that in both cases it is determined by the waiting time distribution. As we shall see this is no more true for the other measure of quantum non-Markovianity.

In the case $\boldsymbol{\lambda} = (1 - \nu, 0, 0, \nu)$, with $0 \leq \nu \leq 1$, one has a mixture of phase-flip and identity, leading to $\boldsymbol{\mu} = (1, 1 - 2\nu, 1 - 2\nu, 1)$ and therefore $\boldsymbol{\lambda}(t) = (1, \lambda_{1-2\nu}(t), \lambda_{1-2\nu}(t), 1)$, with $\hat{\lambda}_{1-2\nu}(u)$ as in Eq. (19). The resulting map again describes pure dephasing, but the off-diagonal matrix element is multiplied by a factor $\lambda_{1-2\nu}(t)$. In order to assess non-Markovianity one now has to study $\lambda_{1-2\nu}(t)$ instead of $q(t)$. Considering again the same class of waiting time distributions, one sees that now a threshold appears, so that one can have non-Markovianity only if $\nu > \frac{1}{2}$, as shown in Fig. 2. This shows that indeed non-Markovianity depends on both elements of the couple $\{\mathcal{E}, f(t)\}$, at variance with the classical case and the previous examples.

The criterion of non-Markovianity introduced in [8] relies on the requirement of divisibility of the quantum map in terms of completely positive maps, so that Eq. (3) applies where now each Λ is a completely positive map. To detect the failure of divisibility one can consider the Choi matrix associated to the map $\Lambda(t, s)$, with $t \geq s$, which can be obtained from Eq. (20). For the general case the expression is somewhat cumbersome, and an alternative way is to look at the sign of the coefficients in the master equation in time-convolutionless form corresponding to Eq. (13). For the case $\boldsymbol{\lambda} = (0, \frac{1}{2}, \frac{1}{2}, 0)$ this path has been followed in [9], the master equation reads

$$\frac{d\rho}{dt} = -\frac{1}{4} \left(\frac{\dot{\lambda}_x(t)}{\lambda_x(t)} + \frac{\dot{\lambda}_y(t)}{\lambda_y(t)} - \frac{\dot{\lambda}_z(t)}{\lambda_z(t)} \right) [\sigma_z \rho \sigma_z - \rho] - \frac{1}{2} \frac{\dot{\lambda}_z(t)}{\lambda_z(t)} [\sigma_+ \rho \sigma_- + \sigma_- \rho \sigma_+ - \rho]$$

with $\lambda_z(t) = q(t)$, $\lambda_x(t) = \lambda_y(t) = g(t)$. In this setting one can consider situations in which the measure of non-Markovianity related to the trace distance is zero, since

the map is divisible in the sense of positive maps, but the intermediate maps are not completely positive. This is the case for the convolution of two different exponential waiting time distributions as in Eq. (23), for a suitable choice of the ratio λ_1/λ_2 . Note that for the situation in which all $\lambda_i(t)$ are different one has

$$\begin{aligned} \frac{d\rho}{dt} = & -\frac{1}{4} \left(\frac{\dot{\lambda}_x(t)}{\lambda_x(t)} + \frac{\dot{\lambda}_y(t)}{\lambda_y(t)} - \frac{\dot{\lambda}_z(t)}{\lambda_z(t)} \right) [\sigma_z \rho \sigma_z - \rho] - \frac{1}{2} \frac{\dot{\lambda}_z(t)}{\lambda_z(t)} [\sigma_+ \rho \sigma_- + \sigma_- \rho \sigma_+ - \rho] \\ & + \frac{1}{4} \left(\frac{\dot{\lambda}_x(t)}{\lambda_x(t)} - \frac{\dot{\lambda}_y(t)}{\lambda_y(t)} \right) [\sigma_x \rho \sigma_x - \rho] - \frac{1}{4} \left(\frac{\dot{\lambda}_x(t)}{\lambda_x(t)} - \frac{\dot{\lambda}_y(t)}{\lambda_y(t)} \right) [\sigma_y \rho \sigma_y - \rho], \end{aligned} \quad (25)$$

and the two intermediate channels have opposite coefficients, so that unless $\lambda_x(t) = \lambda_y(t)$, which is the case considered in [9], one is always negative. In this case however one cannot read divisibility from the sign of the coefficients, since the Lindblad operators appearing in them are not linearly independent, indeed Eq. (25) can be written as

$$\begin{aligned} \frac{d\rho}{dt} = & +\frac{1}{4} \left(\frac{\dot{\lambda}_x(t)}{\lambda_x(t)} - \frac{\dot{\lambda}_y(t)}{\lambda_y(t)} - \frac{\dot{\lambda}_z(t)}{\lambda_z(t)} \right) [\sigma_x \rho \sigma_x - \rho] \\ & -\frac{1}{4} \left(\frac{\dot{\lambda}_x(t)}{\lambda_x(t)} - \frac{\dot{\lambda}_y(t)}{\lambda_y(t)} + \frac{\dot{\lambda}_z(t)}{\lambda_z(t)} \right) [\sigma_y \rho \sigma_y - \rho] - \frac{1}{4} \left(\frac{\dot{\lambda}_x(t)}{\lambda_x(t)} + \frac{\dot{\lambda}_y(t)}{\lambda_y(t)} - \frac{\dot{\lambda}_z(t)}{\lambda_z(t)} \right) [\sigma_z \rho \sigma_z - \rho]. \end{aligned} \quad (26)$$

In the present framework we can indeed point to a situation in which, due to a subtle balance, all coefficients in Eq. (26) are positive, thus granting divisibility, but this is no more true for the coefficients of Eq. (25). This situation is depicted in Fig. 3 considering $\boldsymbol{\lambda} = (\frac{1}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{5})$ and a suitable waiting time distribution. Even though this fact can be easily understood from a conceptual point of view, it is useful to stress it by means of an explicit example.

For the case $\boldsymbol{\lambda} = (0, 0, 0, 1)$ of pure dephasing in order to detect non-Markovianity it is convenient to consider the Choi matrices associated to the maps $\Lambda(t+s, t) = \Lambda(t+s, 0)\Lambda^{-1}(t, 0)$, so that according to Eq. (20) we obtain

$$\Lambda'_{kl}(t+s, t) = \text{diag} \left(1 + \frac{q(t+s)}{q(t)}, 0, 0, 1 - \frac{q(t+s)}{q(t)} \right), \quad (27)$$

where the simplicity of the result strongly depends on the convenient choice of basis. Failure of divisibility is then detected when at least one of the coefficients of these collection of matrices depending on two temporal indexes becomes negative. The sign of the smallest non zero eigenvalue is plotted in Fig. 3 for a waiting time given by the convolution of two identical exponential distributions. In accordance to the result obtained relying on the criterion based on trace distance distinguishability, the map is indeed non-Markovian. Note however that the divisibility property does depend on the initial time considered, i.e. for certain time windows, which obviously include the initial time $t = 0$, the maps $\Lambda(t+s, t)$ are completely positive for any time interval s . At the same time the violation of complete positivity does decrease for large s . Moreover while the two criteria agree in labelling the map as non-Markovian, as discussed in detail in [9] they assign to it different measures. In particular as follows from Eq. (27) the approach

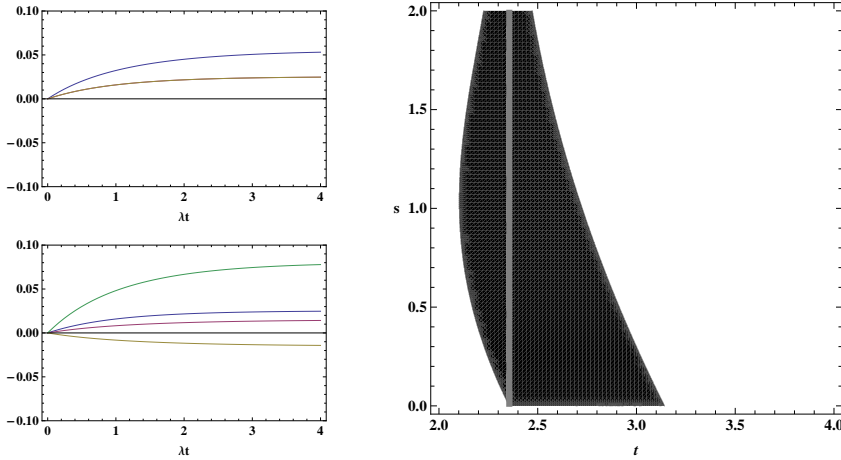


Figure 3. (Color online) (top left) Plot of the two distinct coefficients of Eq. (26). (bottom left) Plot of the four distinct coefficients of Eq. (25). In the latter case one is always negative. The coefficients of the two equivalent expressions of the time-convolutionless master equation associated to the map Eq. (21) for the case $\lambda = (\frac{1}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{5})$, so that $\mu = (1, \frac{1}{5}, -\frac{1}{5}, -\frac{1}{5})$ are obtained from Eq. (17). The considered waiting time distribution is of the form Eq. (23) with $\lambda_2 = r\lambda$ and $r = 0.13$. (right) Plot of the sign of the lowest non zero eigenvalue of the Choi matrix corresponding to the map $\Lambda(t+s, t)$, for a stochastic map corresponding to pure dephasing and waiting given by the convolution of two equal exponentials. The black regions correspond to a negative sign. For certain values of t , corresponding to the points in which the inverse does not exist, in the plot denoted by a gray vertical line, the memory is arbitrary long, while it stays finite for other values.

based on divisibility assigns an infinite measure to this map as soon as the quantity $q(t)$ goes through zero, so that at variance with Eq. (22) processes with different memories are put on the same footing. Let us note that this difficulty can be overcome by keeping divisibility of the time evolution in terms of completely positive maps as a signature of Markovianity, quantifying however its violation in a different way. As suggested in [34] one can introduce a different weight, considering the integral of the arcotangent of the sum of the negative eigenvalues of the matrix Eq. (27) in the time regions in which complete positivity breaks down, renormalizing by the extension of these regions. For the case at hand indeed this modification makes the non-Markovianity measure based on divisibility finite, even though the expression Eq. (22) for the measure based on distinguishability remains much easier to evaluate.

4. Conclusions

In the present manuscript we have considered a class of quantum dynamics which can be obtained from a generic unital stochastic map on \mathbb{C}^2 and a classical waiting time distribution, thus extending the class of examples considered in Ref. [9] to show how the recently introduced notions of quantum non-Markovianity relate to the classical one. The considered examples show how versatile the class of semi-Markov processes

and their quantum counterpart can be in order to study the notion of non-Markovian process in the quantum framework and highlight different possible behavior.

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References

- [1] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002)
- [2] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, J. Math. Phys. **17**, 821 (1976)
- [3] G. Lindblad, Comm. Math. Phys. **48**, 119 (1976)
- [4] A. Barchielli and M. Gregoratti, *Quantum Trajectories and Measurements in Continuous Time*, Vol. 782 of *Lect. Notes in Phys.* (Springer, Berlin, 2009)
- [5] G. Lindblad, Comm. Math. Phys. **65**, 281 (1979)
- [6] L. Accardi, A. Frigerio, and J. T. Lewis, Publ. RIMS Kyoto **18**, 97 (1982)
- [7] H.-P. Breuer, E.-M. Laine, and J. Piilo, Phys. Rev. Lett. **103**, 210401 (2009)
- [8] A. Rivas, S. F. Huelga, and M. B. Plenio, Phys. Rev. Lett. **105**, 050403 (2010)
- [9] B. Vacchini, A. Smirne, E.-M. Laine, J. Piilo, and H.-P. Breuer, New J. Phys. **13**, 093004 (2011)
- [10] H.-P. Breuer and B. Vacchini, Phys. Rev. Lett. **101**, 140402 (2008)
- [11] H.-P. Breuer and B. Vacchini, Phys. Rev. E **79**, 041147 (2009)
- [12] C. W. Gardiner, *Quantum Noise* (Springer, New York, 1991)
- [13] D. T. Gillespie, Am. J. Phys. **66**, 533 (1998)
- [14] M. Lax, Phys. Rev. **172**, 350 (1968)
- [15] R. Dümcke, J. Math. Phys. **24**, 311 (1983)
- [16] A. A. Budini, Phys. Rev. A **69**, 042107 (2004)
- [17] H.-P. Breuer, Phys. Rev. A **75**, 022103 (2007)
- [18] B. Vacchini, Phys. Rev. A **78**, 022112 (2008)
- [19] A. A. Budini and H. Schomerus, J. Phys. A: Math. Gen. **38**, 9251 (2005)
- [20] C. A. Fuchs and J. van de Graaf, IEEE Trans. Inf. Th. **45**, 1216 (1999)
- [21] A. S. Holevo, *Probabilistic and statistical aspects of quantum theory* (North-Holland, Amsterdam, 1982)
- [22] G. Ludwig, *Foundations of quantum mechanics*. (Springer-Verlag, New York, 1983)
- [23] L. Lanz, B. Vacchini, and O. Melsheimer, J. Phys. A: Math. Theor. **40**, 3123 (2007)
- [24] B. Vacchini, in *Theoretical Foundations of Quantum Information Processing and Communication*, E. Bruening and F. Petruccione eds. (Springer, Berlin, 2010), Lect. Notes in Phys. 787, p. 39
- [25] P. Hänggi and H. Thomas, Z. Phys. B **26**, 85 (1977)
- [26] E.-M. Laine, J. Piilo, and H.-P. Breuer, Phys. Rev. A **81**, 062115 (2010)
- [27] J. Dajka, J. Luczka, and P. Hänggi, Phys. Rev. A **84**, 032120 (2011)
- [28] B. Vacchini, J. Mod. Opt. **51**, 1025 (2004)
- [29] B. Vacchini and K. Hornberger, Phys. Rep. **478**, 71 (2009)
- [30] W. Feller, *An introduction to probability theory and its applications. Vol. II* (John Wiley & Sons Inc., New York, 1971)
- [31] C. King and M. B. Ruskai, IEEE Trans. Inf. Th. **47**, 192 (2001)
- [32] E. C. G. Sudarshan, P. M. Mathews, and J. Rau, Phys. Rev. **121**, 920 (1961)
- [33] D. R. Cox and H. D. Miller, *The theory of stochastic processes* (John Wiley & Sons Inc., New York, 1965)
- [34] S. C. Hou, X. X. Yi, S. X. Yu, and C. H. Oh, Phys. Rev. A **83**, 062115 (2011)